Spring 2008

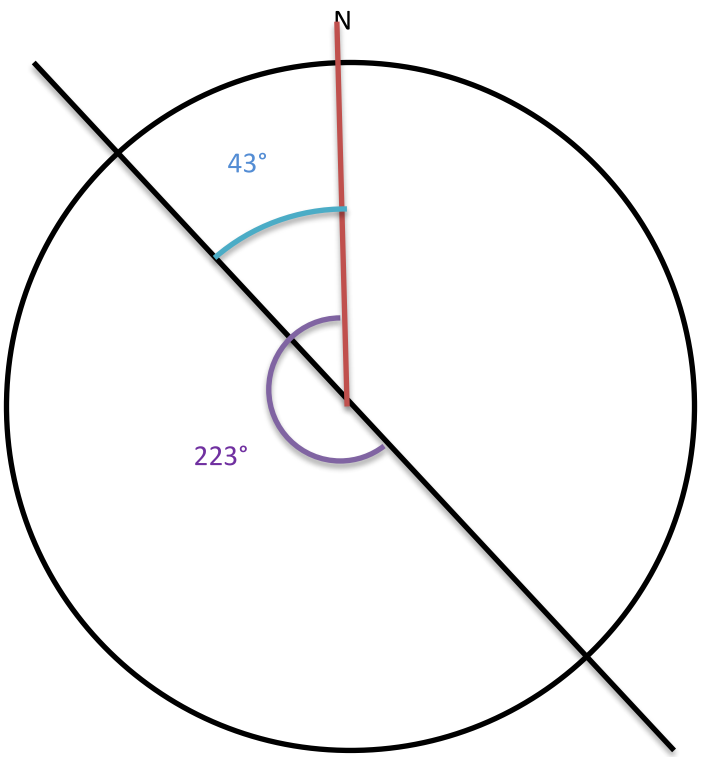
1.

$g(x-1) = x^2 + 1$

$g(2) = g(3-1) = 3^2 + 1 = 10$

So the answer is **E. 10**

2.



Opposite angles are 180 degrees apart, so the runway is also (223 - 180 = 43)° from North, and so has a label of 4

So the answer is **A. 4**

3.

$a^3 + b^3 + c^3 = 2008$

This problem is much easier for a calculator to solve by trial and error than to do by hand. I am going to outline the steps I would take using a TI-83 Plus, as that is the kind of calculator that can be borrowed from the TCC SE math lab.

First we need one variable by itself

$a^3 + b^3 + c^3 = 2008$

$a^3 = 2008 - b^3 - c^3$

$a = \sqrt[3]{2008 - b^3 - c^3}$

Now we have a as a function of 2 variables. This is unfortunate because our calculator only wants functions of one variable. So we will have to take a guess on c. We can start with 1 and go up. Since the TI-83 can show 2 functions in a table without scrolling over, we can take 2 guesses at a time. Replace a with Y, b with X, and c with our guess, like so.

$Y_1= {}^3\sqrt{}$(2008-X^3-1^3)
$Y_2={}^3\sqrt{}$(2008-X^3-2^3)

Hit 2nd and GRAPH to get to the table and look for a whole number in either Y_1 or Y_2.

We should see Y_2=10 when X=10. However the problem states that are values should be distinct, so this solution does not work. So try the next 2 values.

$Y_1= {}^3\sqrt{}$(2008-X^3-3^3)
$Y_2= {}^3\sqrt{}$(2008-X^3-4^3)

Now our table shows that Y_2=12 when X=6 (and 6 when X=12). Since Y_2 had c as 4, our numbers are 4, 6, and 12.

They ask for $a + b + c$.

$4 + 6 + 12 = 22$

So the answer is **B. 22**

4.

For $x^2 + bx + 16$ to factor over the integers, it is necessary and sufficient that b is the sum of 2 numbers that multiply to 16. This gives us the following possibilities.

$16 + 1 = 17$

$-16 + (-1) = -17$

$8 + 2 = 10$

$-8 + (-2) = -10$

$4 + 4 = 8$

$-4 + (-4) = -8$

This gives us 6 values for b that lead to an integer factorization.

So the answer is **E. 6**

5.

$f(x) = x^2 - 2x + 4$

$f(2y) = (2y)^2 - 2(2y) + 4$

$= 4y^2 - 4y + 4$

$f(x) - f(2y) = x^2 - 2x + 4 - 4y^2 + 4y - 4$

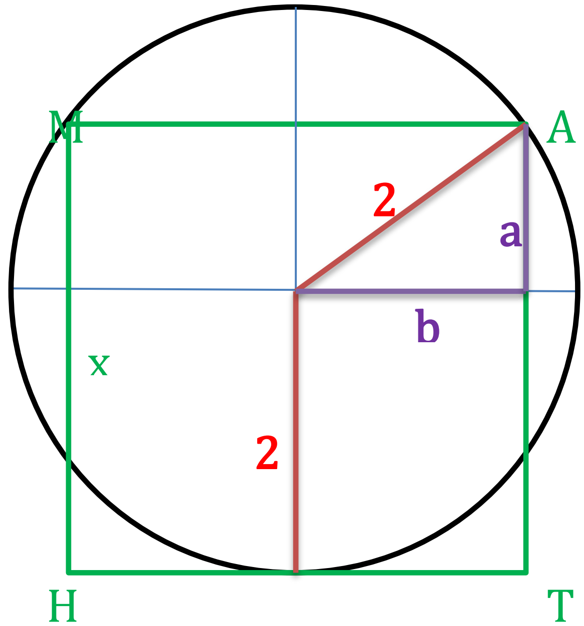
$= x^2 - 4y^2 - 2x + 4y$

$= (x + 2y)(x - 2y) - 2(x - 2y)$ Factor by grouping

$= (x + 2y - 2)(x - 2y)$

So the answer is **D.** $\boldsymbol{x + 2y - 2}$

6.



Using the figure as a reference, with x being the length of any side of the square,

$a^2 + b^2 = 20^2 = 400$	Pythagorean Theorem
$x = MA = 2b$	The shape is symmetrical
$x = AT = 20 + a$	
$2b = 20 + a$	both equal to x
$b = 10 + \frac{a}{2}$	divide by 2
$a^2 + \left(10 + \frac{a}{2}\right)^2 = 400$	substitution for b
$a^2 + 100 + 10a + \frac{a^2}{4} = 400$	FOIL
$\frac{5a^2}{4} + 10a - 300 = 0$	combine like terms into standard quadratic form
$a^2 + 8a - 240 = 0$	divide by $\frac{5}{4}$ (same as multiplying by $\frac{4}{5}$)
$(a + 20)(a - 12) = 0$	factor
$a = \{-20, 12\}$	
$a = 12$	length is not negative
$x = 20 + a = 20 + 12 = 32$	

So the answer is **E. 32**

7.

The coin has to have 2 A's and 1 M in its 3 flips. There are 8 possible arrangements of 3 flips $(2 * 2 * 2)$ and 3 of them have 2 A's and 1 M (MAA, AMA, AAM). This gives a probability of $\frac{3}{8}$.

We need one of each letter from the die. The die has 27 possible outcomes $(3 * 3 * 3)$, 6 of which have 1 of each letter (CTY, CYT, TCY, TYC, YCT, YTC). This gives a probability of $\frac{6}{27}$, or $\frac{2}{9}$ if we reduce.

To find the overall probability, we multiply the individual probabilities. $\frac{2}{9} * \frac{3}{8} = \frac{6}{72} = \frac{1}{12}$

So the answer is **E.** $\mathbf{\frac{1}{12}}$

8.

$$(\log_{624} 625)(\log_{623} 624) \ldots (\log_6 7)(\log_5 6)$$

$$= \left(\frac{\log 625}{\log 624}\right)\left(\frac{\log 624}{\log 623}\right)\left(\frac{\log 623}{\log 622}\right) \ldots \left(\frac{\log 7}{\log 6}\right)\left(\frac{\log 6}{\log 5}\right)$$

$$= \frac{\log 625}{\log 5}$$

$$= \log_5 625 = 4$$

So the answer is **C. 4**

9.

A	M	A	T	Y	C	A	M	A	T
Y	C	A	M	A	T	Y	C	A	M
A	T	Y	C	A	M	A	T	Y	C
A	M	A	T	Y	C	A	M	A	T
Y	C	A	M	A	T	Y	C	A	M
A	T	Y	C	A	M	A	T	Y	C
A	M	A	T	Y	C	A	M	A	T
Y	C	A	M	A	T	Y	C	A	M
A	T	Y	C	A	M	A	T	Y	C
A	M	A	T	Y	C	A	M	A	T

100 divided by 6 is 16 with a remainder of 4, so AMATYC is written 16 times and AMAT is written once. There are then 34 A's. So the probability of 3 A's without replacement is $\frac{34}{100} * \frac{33}{99} * \frac{32}{98} = .037 = 3.7\%$

So the answer is **B. 3.7%**

10.

There are $\frac{5*4}{2} = 10$ combinations of the seniors. So we need at least $\frac{600}{10} = 60$ combinations from the juniors. n juniors results in $\frac{n(n-1)(n-2)}{3!}$ combinations. Thus $\frac{[n(n-1)(n-2)]}{3!} \geq 60$

$n(n-1)(n-2) \geq 60(6)$

$n^3 - 3n^2 + 2n - 360 \geq 0$ FOIL and simplify, subtract 360 from both sides

This does not factor evenly, and even if it could, it would not be easy to find the factorization (the number of possible rational roots is overwhelming). So test the given values. I start with 8 to split the data, allowing for less tests.

$8^3 - 3(8)^2 + 2(8) - 360$

$512 - 196 + 16 - 360$

-28

8 is too low, but just barely, so try 9.

$9^3 - 3(9)^2 + 2(9) - 360$

$729 - 243 + 18 - 360$

144, which is greater than 0.

So the answer is **D. 9**

11.

First let's write an equation for this occurrence, relating the time of the trip compared to the normal time. Time $= \frac{\text{Distance}}{\text{Speed}}$. Driving halfway to work takes $\frac{1}{2} * \frac{D}{S}$. back home at 8 mph faster takes $\frac{1}{2}\frac{D}{S+8}$, and to work at 6 more mph (S+14) takes $\frac{D}{S+14}$. Altogether, this is supposed to take 1.67 times as long as a normal trip, or $1.67\frac{D}{S}$.

$\frac{1}{2} * \frac{D}{S} + \frac{1}{2} * \frac{D}{S+8} + \frac{D}{S+14} = 1.67\frac{D}{S}$

$\frac{1}{2} * \frac{D}{S+8} + \frac{D}{S+14} = 1.17\frac{D}{S}$ subtract ½ D/S from both sides.

$\frac{1}{2(S+8)} + \frac{1}{S+14} = \frac{1.17}{S}$ divide both sides by D, simplify.

$S(S+14) + 2S(S+8) = 2.34(S+14)(S+8)$ multiply by LCD (2S(S+14)(S+8)).

$S^2 + 14S + 2S^2 + 16S = 2.34(S^2 + 22S + 112)$ distribute right, FOIL left.

$3S^2 + 30S = 2.34S^2 + 51.48S + 262.08$ simplify

$.66S^2 - 21.48S - 262.08 = 0$ set equal to 0.

Using the quadratic formula: $S = \frac{21.48 \pm \sqrt{(21.48)^2 - 4(.66)(-262.08)}}{2(.66)}$

$= \frac{21.48 \pm \sqrt{461.3904 + 691.8912}}{1.32}$

$= \frac{21.48 \pm \sqrt{1153.2816}}{1.32}$

$= \frac{21.48 \pm 33.96}{1.32}$

$= \frac{55.44}{1.32}, -\frac{12.48}{1.32}$

$= 42, -9\frac{5}{11}$

Speed is positive, so we can eliminate $-9\frac{5}{11}$

So the answer is **42**

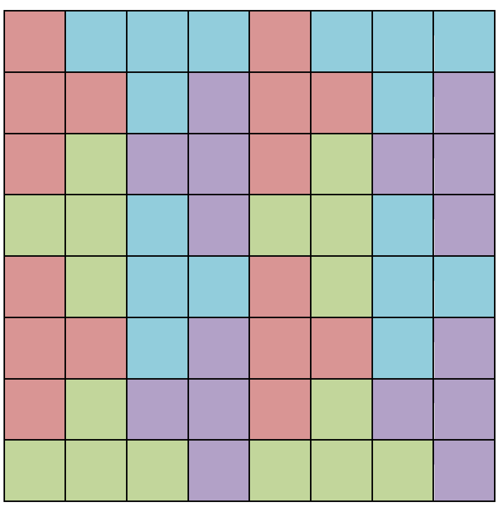
12.

12 and 18 are both multiples of 6, so multiples of them added together will always be a multiple of 6, which 1000 is not. 1000 is 4 more than a multiple of 6 ($996 = 6 * 166$), as is 22. So we need 1 22-pound bag. That leaves us 978 pounds. 978 is 6 more than a multiple of 12 ($972 = 12 * 81$), so we need 1 18-pound bag. that leaves 960 pounds, which can be done with 80 12-pound bags. 80 12-pound bags, 1 18-pound bag, and 1 22-pound bag add up to 82 bags.

So the answer is **C. 82**

13.

Let's start in the top left corner with the 3 block edge vertical, and attempt to add as many blocks as possible with that orientation. The top and bottom edges each require 2 horizontal blocks to remain flat, but otherwise the space can be filled with vertical blocks.



So the answer is **C. 4**

14.

First we need to find 3 biprimes in a row. The smallest biprime is 6, then 10, 14, 15, 21, 22, 26, 33, 34, 35…

So our biprimes are 33, 34, and 35. The largest prime factor of their product is the largest prime factor between them, which is 17.

So the answer is **B. 17**

15.

To maximize the number of green counters, we want pairs of the same color to both be green in all cases, and reds only appearing between greens. If we have 3 greens between each pair of reds, we'll have 2 pairs different and 2 pairs the same for each set. So we start with:

1 1 2 3 2 4 5 6 3 7 8 9 4 10 11 12 5 13 14 15 6 16 17 18 7 19 20 21 8

This has 14 pairs the same and 14 different. But if we add 2 greens at each end, we'll have 16 of each type of pair and still match.

1 2 1 3 4 5 2 6 7 8 3 9 10 11 4 12 13 14 5 15 16 17 6 18 19 20 7 21 22 23 8 24 25

There are 25 green counters, and no way to add more.

So the answer is **D. 25**

16.

$$1.5 > \frac{b}{11} > 1.8$$

$16.5 < b < 19.8$

$b = 17, 18, or\ 19$

$1.5 < \frac{c}{15} < 1.8$

$22.5 < c < 27$

$c = 23, 24, 25, or\ 26$

$1.5 < \frac{c}{b} < 1.8$

The lowest possible value for $\frac{c}{b}$ is $\frac{23}{19} \approx 1.21$. The highest value is $\frac{26}{17} \approx 1.53$

$\frac{25}{17} \approx 1.47$ and $\frac{26}{18} = 1.\overline{44}$ are the next smallest values and are too small. So b=17, c=26

$b + c = 17 + 26 = 43$

So the answer is **A. 43**

17.

We can treat this as a system of three equations in three variables ($r, s, t \in \mathbb{Z} \geq 0$)

$rs + t = 24$

$r + st = 24$

$r + s + t = 25$

Adding the first two equations gives us: $r + rs + t + st = 48$

$r(1 + s) + t(1 + s) = 48$

$(r + t)(1 + s) = 48$

From the third equation: $r + t = 25 - s$

$(25 - s)(1 + s) = 48$

$25 + 24s - s^2 = 48$

$s^2 - 24s + 23 = 0$

$(s - 1)(s - 23) = 0$

$s = 1 \; or \; s = 23$

If $s = 1$, then the first three equations both become $r + t = 24$

This gives us 25 solutions ($r = 0,1,2,3,4, \dots ,24 \; with \; t = 24,23,22,21,20, \dots ,0$)

If $s = 23$, the equations become:

$23r + t = 24$

$r + 23t = 24$

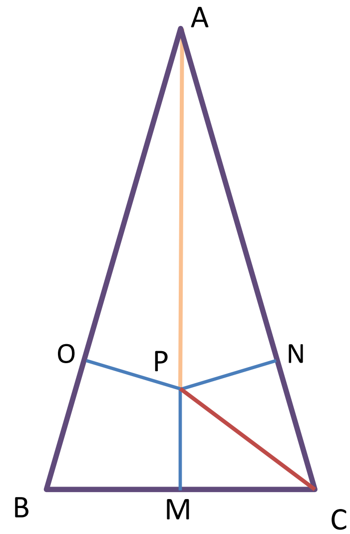
$r + t = 2$

Which can be easily solved to get $(r, s, t) = (1, 23, 1)$.

This gives us a total of 26 solutions.

So the answer is **D. 26**

18.



$\overline{AB} = \overline{AC} = 25$

$\overline{BC} = 14$

$\overline{PM} = \overline{PN} = \overline{PO} = x$

Since $\overline{PM} = \overline{PN}$ and $\overline{PC}$ is shared in ΔPMC and ΔPNC,

$\overline{MC} = \overline{NC} = 14 \div 2 = 7$

Therefore $\overline{AN} = 25 - 7 = 18$

Since ΔAMC is a right triangle, $\overline{CM}^2 + \overline{AM}^2 = \overline{AC}^2$

$7^2 + \overline{AM}^2 = 25^2$

$\overline{AM} = \sqrt{625 - 49} = \sqrt{576} = 24$

Triangle ANP is a right triangle because PN and AC are perpendicular, so $\overline{AN}^2 + \overline{PN}^2 = \overline{AP}^2$

$\overline{AN}^2 + \overline{PN}^2 = (\overline{AM} - \overline{PM})^2$

$18^2 + x^2 = (24 - x)^2$

$324 + x^2 = 576 - 48x + x^2$

$48x = 576 - 324$

$48x = 252$

$x = \frac{252}{48} = \frac{21}{4}$

So the answer is **D.** $\mathbf{\frac{21}{4}}$

19.

This one is more time-consuming than it is difficult. First write out all of the 3 digit perfect squares:

100	225	400	625	900
121	256	441	676	961
144	289	484	729	
169	324	529	784	
196	361	576	841	

Now eliminate those with a repeated digit or a zero because we only have 1-9 and one of each..

169	289	529	729	961
196	324	576	784	
256	361	625	841	

Now get rid of numbers that have the same digits in a different order because we should have a unique set. (169, 196, 961), (256, 625)

289	529	784
324	576	841
361	729	

Now just look for 3 numbers that cover all the digits:

If 289 is one, that leaves 576 and 361 not sharing digits, but they share a 6, so 289 can't be included.

If 324 is one, that leaves only 576, so 324 is not included.

If 361 is one, that leaves 529, 729, and 784. Of those, 529 and 784 do not share digits, so those are our numbers.

$361 + 529 + 784 = 1674$

So the answer is **E. 74**

20.

$a_0 = a_1 = a_2 = 1$

$a_{n-3}a_n - a_{n-2}a_{n-1} = (n-3)!$

Let's work out a few terms:

$a_0a_3 - a_1a_2 = 0!$

$a_3 - 1 = 1$

$a_3 = 2$

$a_1a_4 - a_2a_3 = 1!$

$a_4 - 2 = 1$

$a_4 = 3$

$a_2a_5 - a_3a_4 = 2!$

$a_5 - 6 = 2$

$a_5 = 8$

$a_3a_6 - a_4a_5 = 3!$

$2a_6 - 24 = 6$

Now a pattern develops. $a_1a_2 = 1 = 1!$

$a_2a_3 = 2 = 2!$

$a_3a_4 = 6 = 3!$

$a_4a_5 = 24 = 4!$

Though we haven't proven it (that would take a lot more work), it seems that $a_na_{n+1} = n!$

So $a_{100}a_{101} = 100!$

100 contains 5^2, 95, 90, 85, and 80 contain 5, for 5^6

The situation is similar for 55-75, 30-50, and 1-25, for a total of 5^{24}

So the answer is **C. 24**